University of Idaho
High School Mathematics Competition 2022

Division II Solutions

Division II, Problem 1

Because of winds, an airplane travels from City A to City B at 600 miles per hour but makes the trip from City B to City A at 400 miles per hour. What would be the average speed over a round trip between City A and City B? Furthermore, explain with algebra why the answer does not depend on the distance between the two cities. (Note: The answer is NOT 500 miles per hour.)

Solution: Let d be the distance between the two cities in miles. Then the trip from A to B takes $d/600$ hours and the trip from B to A takes $d/400$ hours. This makes the average speed

$$\frac{2d}{\frac{d}{600}+\frac{d}{400}} = \frac{2d}{\frac{2d}{1200}+\frac{3d}{1200}} = \frac{2400}{5} = 480.$$

Division II, Problem 2

How many solutions does the equation $\cos 2x = \cos x$ have with $0 \le x \le \pi$?

Solution: By the double angle formula, $\cos 2x = 2\cos^2 x - 1$, so the equation reduces to

$$2\cos^2 x - \cos x - 1 = 0.$$

Factoring, we get

$$(2\cos x + 1)(\cos x - 1) = 0,$$

so $\cos x = 1$ or $\cos x = -1/2$. The solutions are now $x = 0$ and $x = 2\pi/3$.

Division II, Problem 3

Simplify

$$\frac{3^{\frac{1}{\ln 3}} \cdot 4^{\frac{1}{\ln 4}}}{20^{\frac{1}{\ln 20}} \cdot 22^{\frac{1}{\ln 22}}}.$$

Solution: For any number a, we have $a^{\frac{1}{\ln a}} = e \approx 2.718$. One can check this since

$$\ln(a^{\frac{1}{\ln a}}) = \frac{1}{\ln a}\ln a = 1.$$

Therefore, the answer is $e^2/e^2 = 1$.

Division II, Problem 4

Fill in the 12 cells along the edge of a 4×4 table with the numbers 1 through 12 (each used exactly once) so that the numbers along each edge has the same sum.

–	–	–	–
–			–
–			–
–	–	–	–

Solution: There are many solutions. Here is one:

6	11	2	7
3			10
9			4
8	1	12	5

Division II, Problem 5

Put the following numbers in increasing order: 3^{4^5}, 3^{5^4}, 4^{3^5}, 4^{5^3}.

Solution: Taking logarithms, we get

$$4^5 \ln 3 = 1024 \ln 3, \quad 5^4 \ln 3 = 625 \ln 3, \quad 3^5 \ln 4 = 243 \ln 4 = 486 \ln 2, \quad 5^3 \ln 4 = 250 \ln 2.$$

Hence,

$$4^{5^3} < 4^{3^5} < 3^{5^4} < 3^{4^5}.$$

Division II, Problem 6

Two circles are put inside a unit square without overlapping. Find the maximum sum of their radii.

Solution: A maximum is reached if the two circles touches each other and both touch two edges of the square. If the radii of the two circles are x and y, then, $(\sqrt{2}+1)x + (\sqrt{2}+1)y$ equals the diameter of the square. I.e., $(\sqrt{2}+1)(x+y) = \sqrt{2}$, which gives $x + y = \frac{\sqrt{2}}{\sqrt{2}+1} = 2 - \sqrt{2}$.

Division II, Problem 7

Let

$$A = 1 + 3 + 3^2 + 3^3 + \cdots + 3^{2022}.$$

Find the remainder when A is divided by 40.

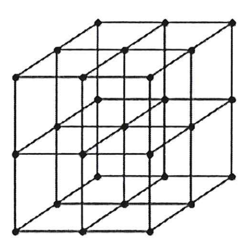
Solution: Notice that $1 + 3 + 3^2 + 3^3 = 40$, so $3^a + 3^{a+1} + 3^{a+2} + 3^{a+3}$ is always a multiple of 40. Now we have

$$A = 1 + 3 + 3^2 + (3^3 + \cdots + 3^6) + \cdots + (3^{2019} + 3^{2020} + 3^{2021} + 3^{2022}).$$

Everything inside some parentheses is a multiple of 40, so the remainder is $1 + 3 + 3^2 = 13$.

Division II, Problem 8

An ant is climbing along the lattice of bars drawn below. It starts from the bottom left front corner and goes to the top right back corner, and always goes up, right, or back. (This means that it always climbs on exactly 6 different bars.) How many different paths can this ant take? (Paths are considered the same if they use the same bars.)

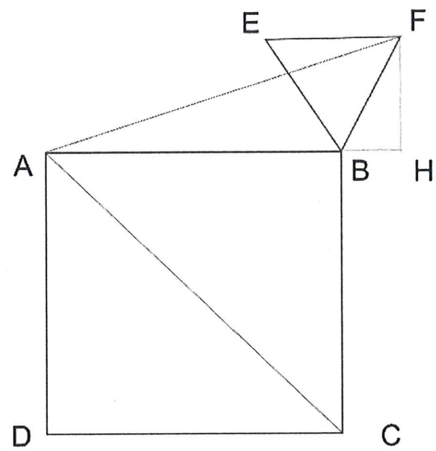


Solution: The ant takes 6 steps, 2 of which are up, 2 of which are right, and 2 of which are back. The paths are determined precisely by which 2 of the 6 steps the ant uses to go up and which 2 of the remaining 4 steps the ant uses to go right. Hence the answer is

$$\binom{6}{2} \times \binom{4}{2} = 90.$$

Division II, Problem 9

Let $ABCD$ be a square and BEF an equilateral triangle as in the picture below. Assume that $EF \parallel AB$ and $AC = AF$. Compute $\tan(\angle BAF)$.



Solution: Let A and a be the length of the square's edges and triangle's edges, respectively. Then $FH = \frac{\sqrt{3}}{2}a$, $BH = \frac{a}{2}$, and hence

$$AF^2 = (A + \frac{a}{2})^2 + \frac{3}{4}a^2.$$

From $AC = AF$,

$$2A^2 = (A + \frac{a}{2})^2 + \frac{3}{4}a^2.$$
$$2A^2 = A^2 + Aa + a^2$$
$$A^2 = Aa + a^2$$

Let $x = A/a$,

$$x^2 - x - 1 = 0$$

We obtain $x = \frac{1+\sqrt{5}}{2}$. That is, $A = \frac{1+\sqrt{5}}{2}a$. Therefore,

$$AH = AB + BH = \frac{1+\sqrt{5}}{2}a + \frac{1}{2}a = \frac{2+\sqrt{5}}{2}a.$$

We conclude

$$\tan \angle BAF = \frac{FH}{AH} = \frac{\sqrt{3}}{2+\sqrt{5}}.$$

Division II, Problem 10

Alice and Bob are very bored and decide to play the following game. They have a long piece of wood with 2022 holes, all in a line. At the start of the game, there are 1000 pegs in the first 1000 holes to the left. Each player in turn moves one peg into an empty hole to its right. The player who makes the last move so that all 1000 pegs are in the last 1000 holes to the right wins. Alice goes first. Give a strategy that Bob can use to win the game no matter what moves Alice makes.

Solution: Think of the board as being divided into pairs of holes (whether filled with a peg or not), with Holes 1 and 2 being partners, Holes 3 and 4 partners, and so on. For every move Alice makes, Bob moves the partner of the peg Alice moved to the partner of the hole Alice moved that peg to. For example, if Alice moves the peg in Hole 207 to Hole 1756, then Bob follows by moving the peg in Hole 208 to Hole 1755. Since Bob always has a move, Alice could not win this game, and so Bob will eventually win.