University of Idaho
High School Mathematics Competition 2024

Competition Guidelines

- Students may only use paper and a writing implement. Other aids (such as calculators, phones, laptops, and sliderules) are not allowed.
- The competition consists of 8 questions. Teams will have 90 minutes.
- Each problem will be graded out of 10 points, for a maximum total of 80 points.
- Every solution should contain a succinct explanation of why the answer is correct or how the team arrived at the answer. Even for a correct answer, this explanation will be taken into consideration in the grading.
- Write your team number in the space provided on each problem. (The problems will be split up for grading.) Please DO NOT write any other identifying information (such as your name or school) on any pages turned in.
- If you require additional pages to write your solution to a problem, use blank paper and staple it to the problem page.
- In the highly unlikely event of a tie score, the team that submitted its solutions earlier will be deemed the winner (of the tie).

Division I, Problem 1

Adam and Bob have been hired to type some old handwritten manuscripts into a computer. Adam would take 16 hours to do the whole job by himself, while Bob would take 12 hours to do the whole job by himself. Suppose Adam and Bob worked together for 6 hours, and then Bob left Adam to finish by himself. How much longer did Adam work for to finish the job? Assume that they don't distract each other or duplicate work and each types at the same rate they would if they were typing by themselves.

Division I, Problem 2

Find the sum of the first 98 nonzero digits of 5/7 (expanded as a decimal).

Division I, Problem 3

Find all pairs of positive integers x and y such that $x^2 - y^2 = 2024$.

Division I, Problem 4

Let

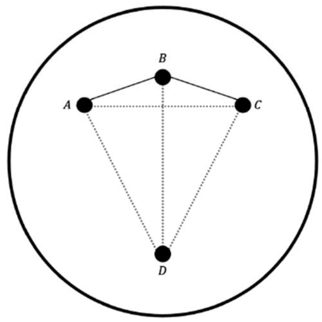
$$x = \sqrt{2024 + \sqrt{2024 + \sqrt{2024 + \sqrt{\cdots}}}}.$$

Express x without any nested square roots.

Division I, Problem 5

You wish to place four candles on a cake so that there are only 2 different distances between any 2 candles. How many different ways are there for you to do this? Make sure to justify that there are only 2 different distances for each way you discover. (Two ways are considered the same if they are similar (in the geometry sense) to each other.) You do NOT need to explain why you have found all the possibilities.

One way of placing the 4 candles is shown in the picture: segments $\overline{AB}$ and $\overline{BC}$ have one length while segments $\overline{AD}$, $\overline{BD}$, $\overline{CD}$, and $\overline{AC}$ all share a second length.



Division I, Problem 6

I have a staircase with 10 stairs. To go up the staircase, I can choose to go up one or two stairs on each step that I take. How many different ways can I climb the staircase?

Note that climbing 1+2+2+1+2+2 is different from 2+2+1+2+1+2 (and there are many other ways of arranging four 2's and two 1's that are all considered different).

Division I, Problem 7

Consider an isosceles triangle with base length 6 and height 4. (The base is NOT one of the two equal sides.) Suppose I have a semicircle with its straight side on the base of the triangle and with its round side tangent to the two other sides of the triangle. What is the radius of this semicircle?

Division I, Problem 8

Zoe plays the following one player game. On the table are 12 envelopes, numbered $1, \ldots, 12$, and the envelope numbered n has $\$n$ inside. On each turn, Zoe can take any envelope as long as that envelope has at least one other envelope labeled by one of its factors still on the table. Then all of the factors of the envelope taken are removed. Zoe then takes another turn, continuing until she can no longer take any turns (because none of the envelopes on the table have any of their factors other than themselves on the table). What is the maximum amount of money Zoe can get?

For example, Zoe can start by taking envelope 6. Once she does so, envelopes 1, 2, and 3 are also removed from the table. On the next turn, Zoe cannot take envelopes 4, 5, 7, 9, or 11 because they have no factors on the table. If Zoe takes envelope 10, then envelope 5 is also removed, and then Zoe has a choice of taking 8 or 12, either of which would end the game.

Division II, Problem 1

Adam and Bob have been hired to type some old handwritten manuscripts into a computer. Adam would take 16 hours to do the whole job by himself, while Bob would take 12 hours to do the whole job by himself. Suppose Adam and Bob worked together for 6 hours, and then Bob left Adam to finish by himself. How much longer did Adam work for to finish the job? Assume that they don't distract each other or duplicate work and each types at the same rate they would if they were typing by themselves.

Division II, Problem 2

Find the smallest integer n with $n > 2024$ such that the equation $x^2 + x - n = 0$ has two integer solutions.

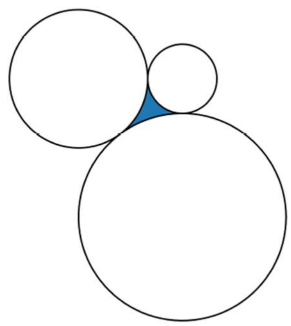
Division II, Problem 3

What is the largest positive integer c such that there are NO positive integers a and b with

$$\frac{a}{14} + \frac{b}{15} = \frac{c}{210}?$$

Division II, Problem 4

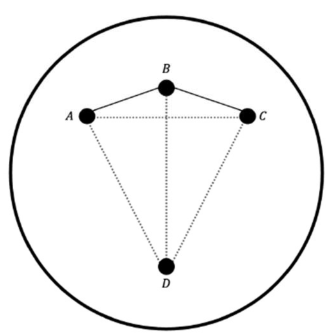
Three non-overlapping disks of radii 1, 2, and 3 are placed so that they are all tangent to each other. What is the area of the region between the three disks? (This is the shaded region in the diagram.)



Division II, Problem 5

You wish to place four candles on a cake so that there are only 2 different distances between any 2 candles. How many different ways are there for you to do this? Make sure to justify that there are only 2 different distances for each way you discover. (Two ways are considered the same if they are similar (in the geometry sense) to each other.) You do NOT need to explain why you have found all the possibilities.

One way of placing the 4 candles is shown in the picture: segments $\overline{AB}$ and $\overline{BC}$ have one length while segments $\overline{AD}$, $\overline{BD}$, $\overline{CD}$, and $\overline{AC}$ all share a second length.



Division II, Problem 6

Solve

$$2^x - 2^{6-x} - 12 < 0.$$

Division II, Problem 7

What is the volume of the largest hemisphere that can be hidden inside a circular cone of radius 5 and height 12? You may assume the bottom of the hemisphere is (part of) the bottom of the cone, and the hemisphere can touch but not pass through the cone.

Division II, Problem 8

Zoe plays the following one player game. On the table are 12 envelopes, numbered $1, \ldots, 12$, and the envelope numbered n has $\$n$ inside. On each turn, Zoe can take any envelope as long as that envelope has at least one other envelope labeled by one of its factors still on the table. Then all of the factors of the envelope taken are removed. Zoe then takes another turn, continuing until she can no longer take any turns (because none of the envelopes on the table have any of their factors other than themselves on the table). What is the maximum amount of money Zoe can get?

For example, Zoe can start by taking envelope 6. Once she does so, envelopes 1, 2, and 3 are also removed from the table. On the next turn, Zoe cannot take envelopes 4, 5, 7, 9, or 11 because they have no factors on the table. If Zoe takes envelope 10, then envelope 5 is also removed, and then Zoe has a choice of taking 8 or 12, either of which would end the game.